Orange Spheres and Odd Even Apples

by Hans Dybkjær
Edited by Robert J. Lang (http://langorigami.com)
crease patterns (/thefold/keywords/229) complex (/thefold/keywords/234) math (/thefold/keywords/237) geometric (/thefold/keywords/244) objects (/thefold/keywords/248) plants (/thefold/keywords/249) design techniques (/thefold/keywords/251) folding techniques (/thefold/keywords/252) video (/thefold/keywords/254)

Somebody asked me, *Can you fold apples and oranges?*, and without blinking I answered, *Of course, you can fold anything!* Afterwards I went home, sat down, and stared at the paper. Flat. As always. And then at the orange sitting innocently next to it. Round as a sphere.

This article tells the tale of the higher spheres of oranges and apples: How I got there, and how to make them.


(/system/files/thefold/fruit.jpg)
Click images for larger versions.

The initial path

The target is round fruit, like apples and oranges: basically spherical objects.

The surface of a sphere curves in all directions. Whereas paper is flat and can only curve in one direction at a time, and even that may require special techniques and diagramming, cf. previous articles in *The Fold* [Hudson 2011, Lang 2013, Gieseking 2013].

Many approaches are possible (see related work at the end), but from the outset I had a foggy idea of making the apple grow organically from the center and out which immediately leads to the use of regular polygons.

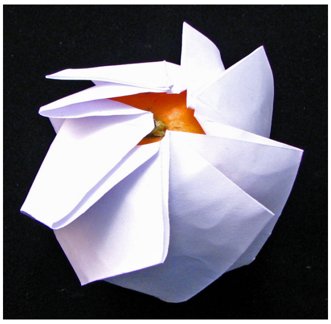

(/system/files/thefold/fruit-section.jpg)

The default origami paper is a square, but that is a coarse approximation to a sphere even if the traditional waterbomb is often presented as a "ball" or a "balloon". Apple and orange flowers have five-fold symmetry; the cross section of oranges have twice as many segments even if my orange didn't seem able to count correctly, and an apple has ten small, green spots from the stamens.

I settled on octagons because they are easy to produce from any rectangle and may give a reasonable circle approximation without producing too many creases (octagon diagram (/system/files/thefold/octagon-plain.svg)). Normal apples are odd numbered, but we are doing even apples—which is an odd choice.

Brute force folding

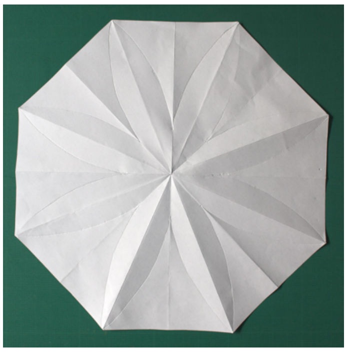
One design technique I have used several times is to crumple the paper to make it malleable, then shape it into the model perceived, and finally study the result to figure out how to replicate using controlled folding.This is an amazingly powerful heuristic design approach. In this case the final shape is given: a sphere. So why not just wrap the octagon around an orange and see what happens?


(/system/files/thefold/wrapped.jpg)

You can see the result in the pictures. It may be difficult to discern from the picture, but the curved lines end up at the corners of the polygon.

The curved lines immediately reminded me of the crease patterns (CPs) of the pots of [Lang 2005-15]. But we need to find out what the curves are, and how to get rid of the unfruity flaps. Of course, we could just exploit the wrapping curves and refine the result, but let's see if we can learn a bit more.

(/system/files/thefold/wrap.jpg)



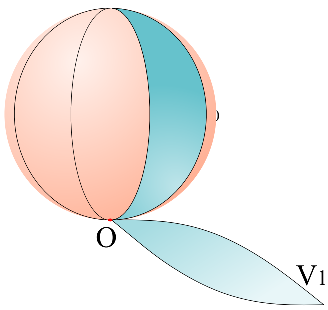
(/system/files/thefold/brute_scored.jpg)
Fruit wrap curve transferred to an octagon, mirrored, and repeated to form 8 gores.

(/system/files/thefold/brute_precreased.jpg)
Precreasing: The space between the gores (see later) has been flat-folded from the back, and unfolded.

(/system/files/thefold/brute_folded.jpg)
And done: Collapsed to form an approximate sphere.

Determining the curves

A curved wrapper line and its mirror together form a *gore*, a pole-to-pole section of a sphere where typically the borders follow lines of constant longitude. With an octagon, eight gores together will make a sphere, each filling 45° (or $\pi/4$ radians). It corresponds to what happens to the surface when you cut an apple into 8 parts or peel an orange to fashion a flower. Our task is to find out what the gores are and how to approximate them for folding.



(/system/files/thefold/peeling-gores.svg)

Since the gores constrain the paper symmetrically along both curved lines, they will curve along their center line and be straight perpendicular to that.

Define the following quantities:

$$O$$

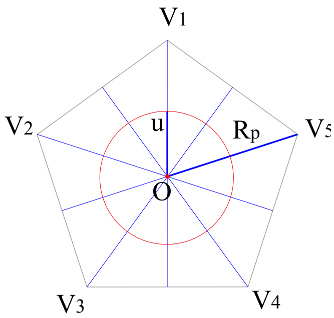
The center or origo of the polygon.

$$N$$

The number of sides.

$$R_p$$

Radius of polygon (radius of circumscribed circle, the distance from center to corner).



(/system/files/thefold/polygon_math.svg)
Polygon with $N = 5$ sides.

We will wrap the polygon such that the center O and the vertices V_j of the polygon will be at opposite points (*poles*) of the sphere. Each polygon radius R_p will lie on a great circle passing through the poles.

$$R = \frac{R_p}{\pi}$$

Radius of the wrapped sphere.

$$u = 0 \ldots R_p$$

Parameter moving along the polygon radius alias the great half circle from pole to pole of the sphere.

$$\phi = \pi - \frac{u\pi}{R_p}$$

The colatitude (the angular distance from the pole).

$$r = R\sin(\phi) = \sin\left(\frac{u\pi}{R_p}\right)\frac{R_p}{\pi}$$

Radius of the latitude circle c at the colatitude ϕ.

$$c = 2\pi r$$

The circumference of the latitude circle.

With all this in place, we can consider how to make the gores. The latitude circle is sandwiched between the inscribed and the circumscribed polygons, and we could use either of the three as the width of the flat gores. We get these side widths:

$$d = \frac{c}{N} = \frac{2\pi r}{N} = \sin\left(\pi - \frac{u\pi}{R_p}\right)\frac{R_p}{2}N$$

The part of c corresponding to one gore.

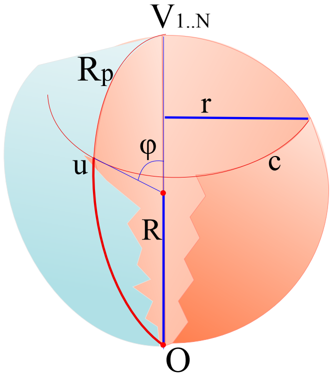
Using d as the width of the flat gores, the circumference of the polygon formed by the gores will equal the length of corresponding latitude circle. Two curves inside a flat gore will have constant longitude. Between these, the gore will lie inside the sphere, and outside these, the gore will be outside the sphere. The equator for $R_p = \pi$ and $N = 8$ is $\frac{\pi}{4}$.

$$s = 2r\sin\left(\frac{\pi}{N}\right) = \sin\left(\frac{u\pi}{R_p}\right)\sin\left(\frac{\pi}{N}\right)\frac{2R_p}{\pi}$$

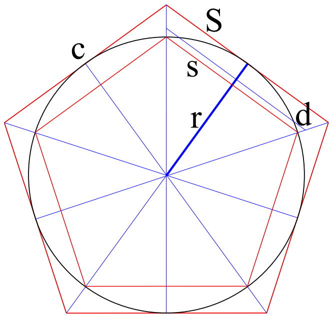
The side of the inscribed polygon of c. Equator for $R_p = \pi$ and $N = 8$ is $2\sin(\pi/8)$, 2.6% smaller than d. Using s will make gores whose sides are longitudes, but whose surface falls strictly inside the sphere.

$$S = 2r\tan\left(\frac{\pi}{N}\right) = \sin\left(\frac{u\pi}{R_p}\right)\tan\left(\frac{\pi}{N}\right)\frac{2R_p}{\pi}$$

The side of the circumscribed polygon. The equator is 5.5% larger than d. The center of the gore has constant longitude, and all of the gore lies outside the sphere.



(/system/files/thefold/sphere-1.svg)
Projection of u onto a sphere corresponds to a colatitude ϕ and defines a latitude circle c.

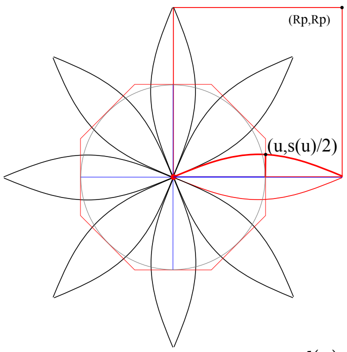


(/system/files/thefold/circle.svg)

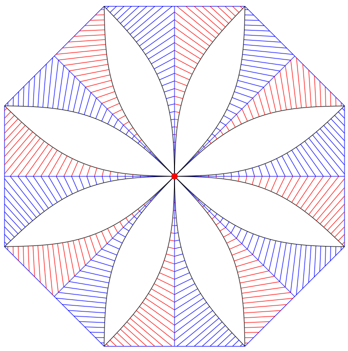
Constructing the gore CP

Drawing the gores is now easy. We plot $\left(u, \frac{d(u)}{2}\right)$ for $u = 0 \ldots R_p$, mirror the curve along the u-axis to obtain a full gore, and duplicate and rotate to obtain all N gores matching the polygon corners.

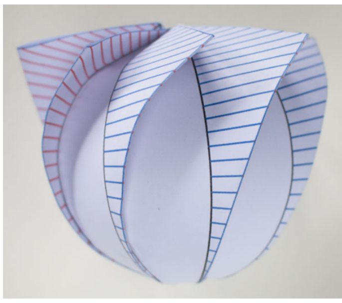
If we only draw one curve of each gore, we get the CP for the flapping wrapper. Score along the curve with the red gradients, crease the center axes, and then fold the flapped near perfect ball.



(/system/files/thefold/plot.svg) Plot of $d(u)$ (*circle plot*), mirrored and repeated 8 times.



(/system/files/thefold/gradients.svg) Circle plot with gradient lines perpendicular to the curves.



(/system/files/thefold/twist_flapped_1980.jpg) Twist-folded circle plot. The lines stick straight out while the paper curves in the other direction.

Getting the flaps inside

The gradients spread out as they extend from the gores, and curving concavely on the inside they clearly need to wiggle. Theoretically that is possible: place the gradients really close together, pleat them, et voila. In all likeliood they will behave like fabric, as when you sew gores together to make a stuffed fabric ball. However, fabric needs sewing to keep the curves together; in origami you need some kind of lock.

Score the curves and gradients in the circle pattern above, and try to fold it with the flaps on the inside. Do this by precreasing from the backside (to become inner side), folding the gradients flat to the side. They will need to pleat, and you quickly realize that it is futile to try to fold all the drawn gradients. Instead, find some "natural" breakpoints along the curve. These breakpoints—and to some extent the overlapping gradients—will serve as the locks we need. Apart from foldability, take care of esthetics, like avoiding a breaking point around the equator.

(/system/files/thefold/folding_gore_sides.jpg)

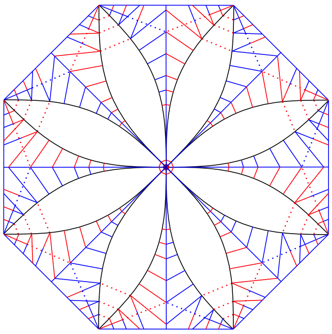
Locking and closing the top is a special problem. Two kinds of creases achieve both. First of all, the entire flap corner is folded behind. Secondly, the curvature of the top is emphasized by marking (softly) three selected orthogonals to the gore.

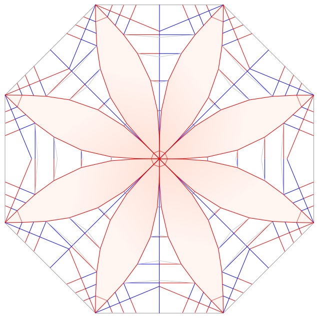

(/system/files/thefold/folding_top_lock.jpg)

Having noted where the creases go, thinning out the unused gradients, and adding the orthogonals, we finally have a crease pattern.

The gore curves are still curved, though in theory flat folding the gradients to the side will make the sections linear. However, by scoring the curves, the outside curvature ends up smoother.

Since the aim of these added creases is to hide away the flaps and lock the sides and top, there is considerable tolerance in the creases. Compare the second crease pattern, having linear gore side sections as well as flap creases that do not align with the gradients. This pattern is inspired by the brute force pattern from the above.


(/system/files/thefold/gradient_cp.svg)
Curved CP.

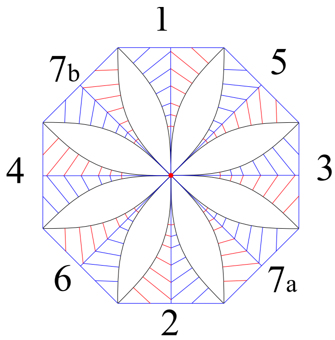

(/system/files/thefold/polygon_orange_cp.svg)
Linear sections CP.

Folding the apple: The difficult stuff

Towards the end of the article you will find a photo sequence, but there are some points of more general interest.

The folding process

The spheres here follow a common sequence of three steps: crease, collapse, shape. This article spends most time on the crease pattern, but it is the collapse and finishing lock and shape steps that require the most skill.

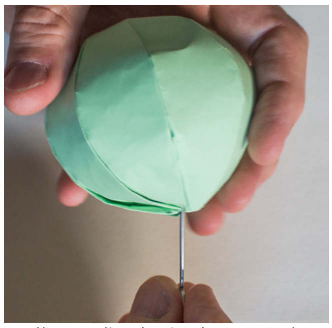


(/system/files/thefold/seam_order.svg)

The seams are best collapsed in a balanced order as illustrated to the right. The last two seams are best taken together, otherwise it is difficult to access the very last one, and doing the last one alone will create tensions straining the first seams unnecessarily.

The hole in the top becomes too small for fingers. I use a flat, but narrow pair of twezers to get these locks into place. Eventually the hole is even too small for tweezers, and I use a modified skewer to fix and smooth dents and seams from the inside.

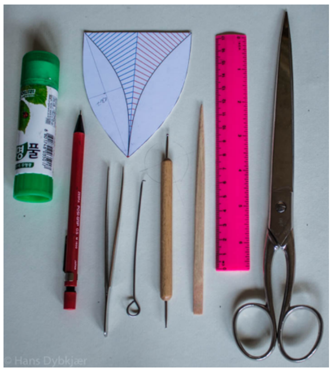
With copy paper and other non-thick papers, precreasing alone works well. However, with thicker paper, and for greater finish, scoring may help. This is particularly true for the gore sides, as these must be precise. Small deviations in gore widths and flap breaks can be seen in the final result. Also, using curved gore sides and scoring along these add to the visual impression of a curved sphere. Using a template helps greatly with precision when you score curves.


(/system/files/thefold/step-18.jpg)

The tools

This is one set of models that cannot be folded without tools. Right to left:

- Scissors: To cut out the octagon, whether it be created by folding or printing.
- Ruler: To check measures, and when scoring.
- Folding bone: With a tip and a flat end. Made from a bamboo chop stick in a few minutes.
- Embossing tool: For scoring, thin end only. Alternatively, the folding bone tip could be used, but the embossing steel tip is more specialized.
- Hook skewer: With a small handle and a hook at the other end. Adapted from a metal skewer. Used for fixing dents etc. when finishing. The hook itself is not used, but the rounded end is good for smoothing and poking the paper, whereas the hook shape enables it to pass through the small hole in the top of the sphere. The handle end makes it easier to hold than a mere skewer.
- Tweezers: To reach inside to fold the final locks. Has flat but narrow ends. Used until the hole becomes too small and the hook skewer takes over.
- Pencil: Markings etc.
- Gore template: For scoring the curved gores. A few layers of tough paper glued together.
- Glue: GLUE? Yes. Used to keep the flaps of the flapped sphere in place. Foil or wet folding might have done the deed (almost) if you think that would be less cheating. All other models, including all apples and oranges, have been made without glue as they actually contain a locking mechanism.


(/system/files/thefold/tools.jpg)

The paper

Since we want curved forms, one might believe soft paper would be good. However, soft paper is bad for the locks applied and for keeping the curve shape. Gores only work for the softer fabric because the seams are sewn together and the fabric ball is stuffed.

The apples and oranges in the pictures below are all folded from patterns 3 and 5. From left to right:

- Aluminum foil sandwiched between 35 gsm green mango paper (a bit stiff) and 30 gsm tissue paper (soft). Spray glue (3M Display Mount). The leaf is from Kraft paper.
- Aluminum foil sandwiched between two layers of 30 gsm tissue paper. Spray glue.
- Green elephant hide 110 gsm.
- Orange copy paper 80 gsm, stiff quality.

Common to all these paper types is they keep a fold well. The aluminum foil increases fold retention, but is soft and crumples easily which we definitely don't want—apples are shiny and smooth. This issue is alleviated by the paper sandwich. However, both the elephant hide and the rather stiff (but standard) quality of copy paper work as well. I have not tried wet folding.

(/system/files/thefold/fruit-folded-2.jpg)

(/system/files/thefold/fruit-folded.jpg)

The organic growth

Already by looking at the crease patterns you feel the organic nature of this model. And indeed it lends itself well for a stop motion version. Done in a simple setup, with some patience, and quickly assembled in iMovie on a MacBook 12".

Enjoy.

Perspectives

Hopefully you have enjoyed this tale from the spheres of oranges and odd even apples. My travel between January 2014 and now has been fun.

(/system/files/thefold/working.jpg)

Though the presentation roughly follows the work progress, real life had a lot of digressions. Just see the snapshot photo from early on where a number of random attempts lie around. One stumbling block was to lock and finish in a satisfying way, another to try to find reasonable landmarks.

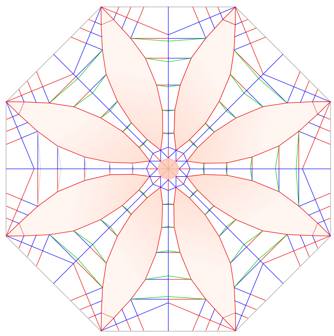
Ending up scoring printed crease patterns, I could just as well have used decagons and been more botanically correct. You could say that eight is chef-correct for cutting an apple into boats. Also, the octogonal version has fewer creases.

The math is just an curious afterthought that may increase the understanding of what happens, and that contributes to optimizing the fold. The ten-minute wrapping approach essentially nailed the crease pattern already.

(/system/files/thefold/decaspheres.jpg) Deca-spheres. Tissue-foil-copy paper sandwich, and plain copy paper.

Many variations are possible, not counting the continuum of possible flap pleatings that do not show up on the outside. One variation has already been seen in the above pictures of apples and oranges, where the oranges have a small knob on the end, obtained by reducing the gore sizes, leaving a space in the center for this. As the gores get more space between them, this also provides better options for shaping the bottom part of the sphere more roundly.

I've got at least two open questions. The first is how to make pears, or any other shape with a concave curve. Though the pear of [Mitani 2011] is purely convex, the vase example of [Gieseking 2013] demonstrates concave shapes using the cylinder technique, and it might also be possible to use the polygon twirl technique of the present paper. However, the real problem is to stuff the excess flaps inside the fruit. The question is if this is possible.



(/system/files/thefold/polygon_orange_with_flower_cp.s

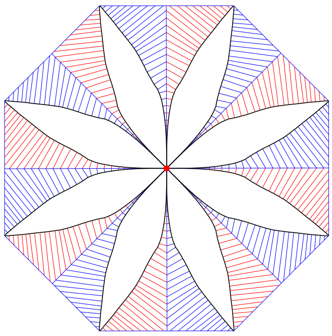
The second is the lack of landmarks. The specimens I ended up with can only be made by scoring the crease pattern on the paper. Some of the specimens on my working table picture are from attempts to free-fold the spheres, but the lack of natural landmarks makes it difficult to get right, and even more difficult to repeat twice in a row the same way. Well, the latter could be an argument that it requires true artistic skills in the same way as wet folding life like animals does. The need for scoring is a fate shared with many other curved folds (and computer constructed patterns), but by approximating the gore curves with line segments it might be possible to find approximate landmarks. But which and how?

Related work

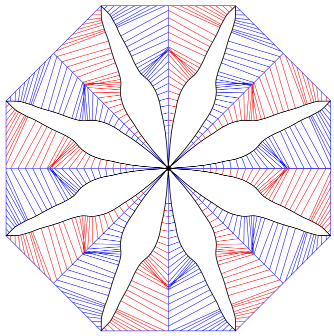
Folding spheres is neither a new nor unique thing. Cartographers have long worked with both how to project the Earth onto paper, as well as reversely, how to draw a map such that it can be glued onto a globe. For printing purposes the gores are usually connected around the equator. For folding, that would imply having two open ends to lock, as well as an entire side

Many examples of rounded models exist, one of which is my own Octopus [Dybkjær 2008], even if it has just a bit more appendices than your average apple.

Mitani [2010, 2011] demonstrates a number of spheres (and pears) folded from cylinders. They look fine, in fact gorgeous, but still have the flaps on the outside, and are made from a cylindric base where I would like something "growing" from the flower bottom.

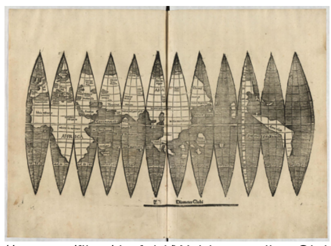


(/system/files/thefold/gradient_pear.svg) Purely convex pear.



(/system/files/thefold/gradient_pear_2.svg) Concave pear. Foldable?

More in line with what I wanted, are some models by Szinger [2008]. A great hot air balloon and other space objects from 2007, and actually a round sphere from 2008 coming even with a crease pattern. It does not, however, provide insight into how one would arrive at this pattern (I still don't comprehend how it is supposed to lock), the gores are not symmetric, and every second gore is divided into four leaving some very visible extra seams. It *is* from a square, but our pattern can trivially be made from a square—the four corners can easily be hidden on the inside when flat-folding the flaps, or, e.g. via rabbit ear folds, be extended to the outside as a short stem. On the other hand, Szinger's pattern may lend itself more easily to merging the curvy folds into other models, as seen, e.g., in his zeppelin or rocket.



(/system/files/thefold/Waldseemuller_Globensegmente_ 2.jpg) Waldseemüller's disrupted gore map, 1507 [[Waldseemüller]].

Chapman-Bell [2009] has made a pattern that is much like our orange, just used for the flap version, and the knob at the end is pointy, not flat. In fact, he has an orange version from 10 gores which appears a bit longer and which in color and shape resembles a lemon (with flaps).

The pots and vases of Lang [2013, 2005-15] share the polygonal twirl symmetry of the spheres here, some are even close to a round shape, but all of them expose the flaps (in highly decorative ways) on the outside.


(/system/files/thefold/szinger-sphere.jpg)
Sphere by Szinger (32-segment version, folded by Hans Dybkjær).

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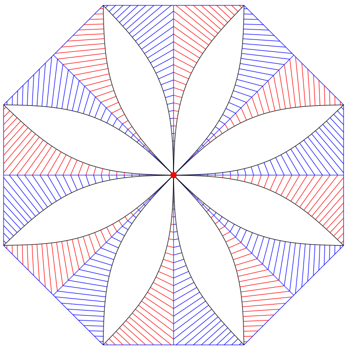
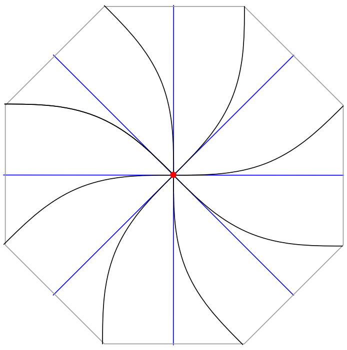
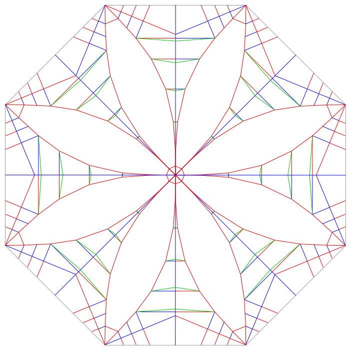
Martin Waldseemüller: Globe version of his 1507 world map where he named America. This version is a later print. http://epub.ub.uni-muenchen.de/13138/ (http://epub.ub.uni-muenchen.de/13138/)

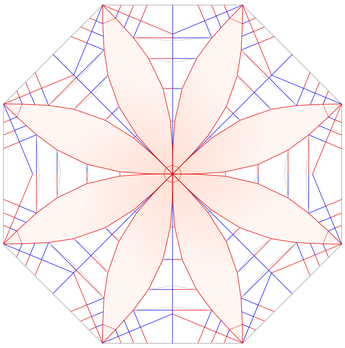
Crease patterns

These are suitable for printing and folding. Click to get a larger version in a separate window, and print directly from your browser. Or download to edit in your favorite vector drawing editor. I use Inkscape (http://inkscape.org).

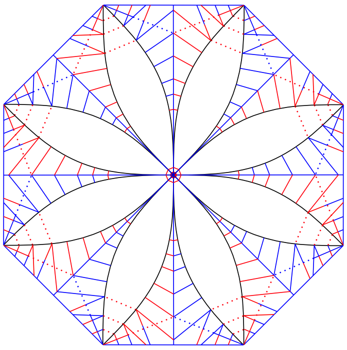

(/system/files/thefold/gradients.svg)
Pattern 1: Gores and gradients, octagon based.


(/system/files/thefold/ideal_twist_cp.svg)
Pattern 2: Minimal pattern for flapped twist sphere.

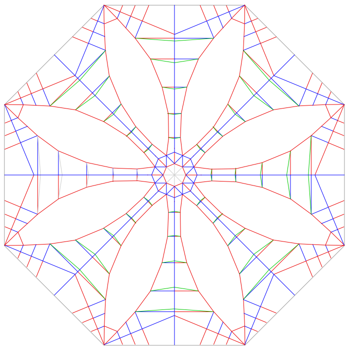

(/system/files/thefold/polygon_cp.svg)
Pattern 3a: First standard pattern, mimicked from brute wrapping fold.



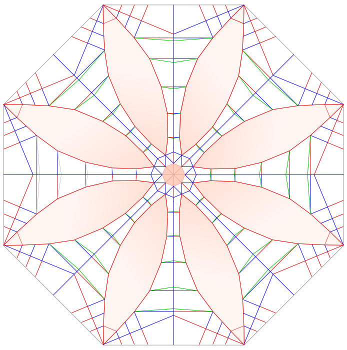
(/system/files/thefold/polygon_orange_cp.svg) Pattern 3b: Coloured for oranges.



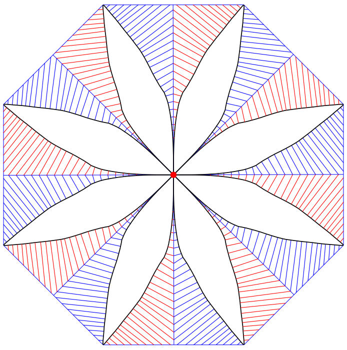
(/system/files/thefold/gradient_cp.svg) Pattern 4: Second standard pattern, based on gradients.



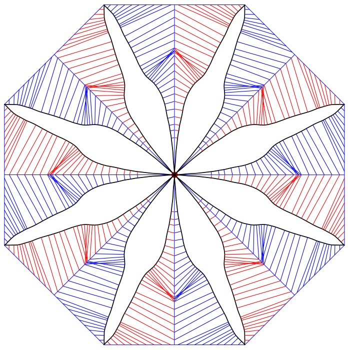
(/system/files/thefold/polygon_with_flower_cp.svg) Pattern 5a: Pattern 3 modified for oranges with a knob.



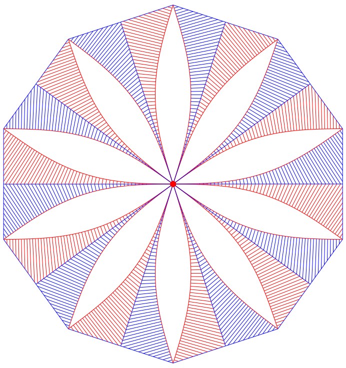
(/system/files/thefold/polygon_orange_with_flower_cp.svg) Pattern 5b: Coloured for oranges.



(/system/files/thefold/gradient_pear.svg) Pattern 6: Convex pear.

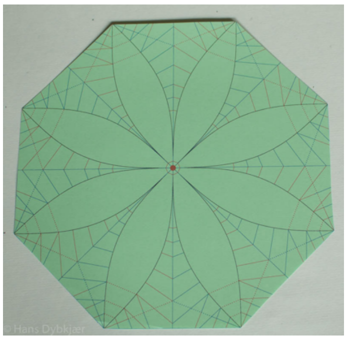


(/system/files/thefold/gradient_pear_2.svg) Pattern 7: Pear with concave interval, probably not foldable.

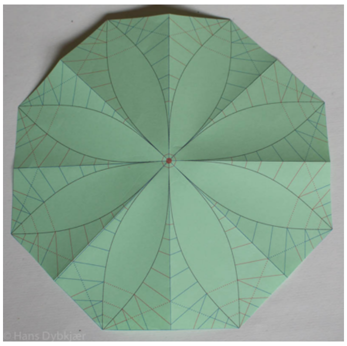


(/system/files/thefold/decagon.svg) Pattern 8: Decagon with flap gradients.

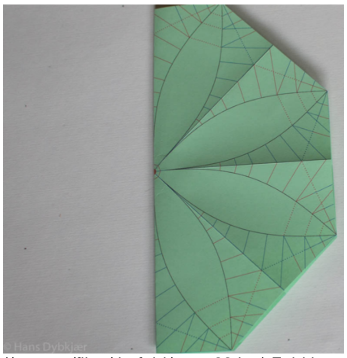
Instructions step-by-step (almost)



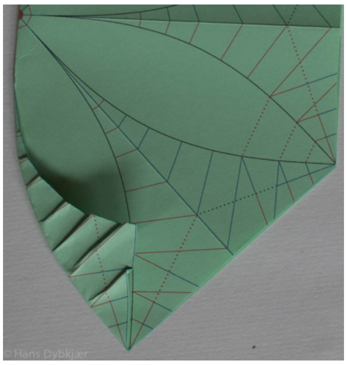
(/system/files/thefold/step-01.jpg) Print or score the pattern on the paper.



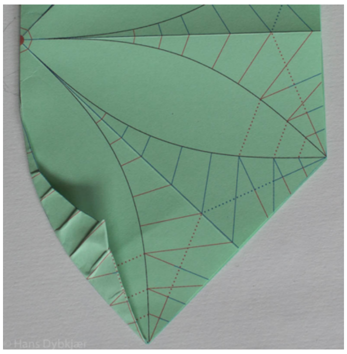
(/system/files/thefold/step-02.jpg) Mark the four axes.



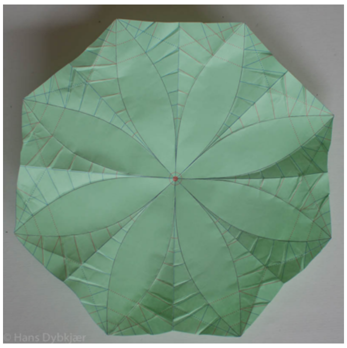
(/system/files/thefold/step-03.jpg) Fold in half.



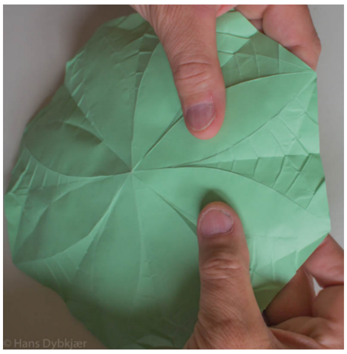
(/system/files/thefold/step-04.jpg) Fold flap along the curve. Number of breaks matters, gradient direction does not.



(/system/files/thefold/step-05.jpg) Fold the corner behind - will lock the gore tips together.



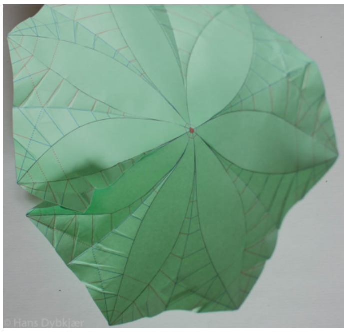
(/system/files/thefold/step-06.jpg) Repeat with all flaps, always to the right. And unfold completely.



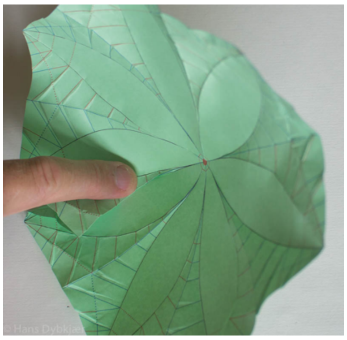
(/system/files/thefold/step-07.jpg) Grab two gores from the outside.

(/system/files/thefold/step-08.jpg) Begin to zip them together - start from the center.

(/system/files/thefold/step-09.jpg) They almost zip all by themselves - all the way to the corner.

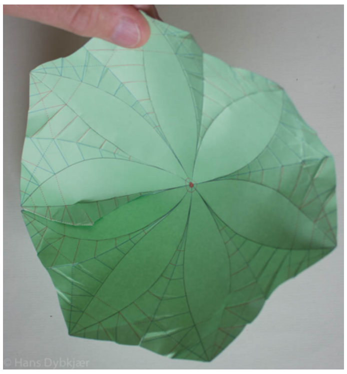


(/system/files/thefold/step-10.jpg) Seen

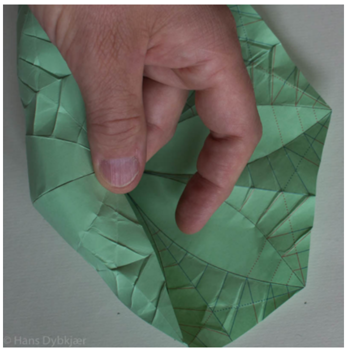


(/system/files/thefold/step-11.jpg) Smooth from the inside.

the seam.



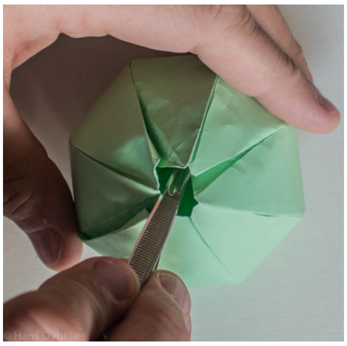
(/system/files/thefold/step-12.jpg) Fold the flap corner and tip lock behind.



(/system/files/thefold/step-13.jpg) Firmly emphasize the curvature of the sphere.

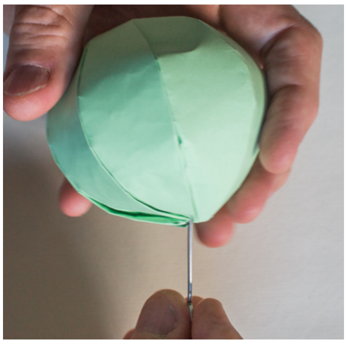
(/system/files/thefold/step-14.jpg) Finished first seam.

(/system/files/thefold/step-15.jpg) Repeat with the remaining 7 seams - make the two last and most difficult ones together.

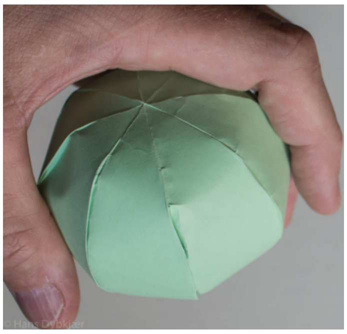


(/system/files/thefold/step-16.jpg) Use tweezers to lock the two last flaps on the inside, fix the others, and help the curvature.

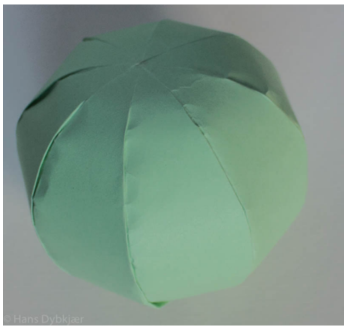
(/system/files/thefold/step-17.jpg) There may be some dents.



(/system/files/thefold/step-18.jpg) Use a thin skewer to mend the dents.



(/system/files/thefold/step-19.jpg) Like this.



(/system/files/thefold/step-20.jpg) The

finished sphere.

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